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TOTALLY REAL SUBMANIFOLDS IN A KAEHLER MANIFOLD

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1. Introduction

Let $\bar{M}$ be a Kaehler manifold of dimension $2(n+p)$, $p \geq 0$, and M an n-dimensional Riemannian manifold. Let J be the complex structure of $\bar{M}$. We call M a *totally real* submanifold of $\bar{M}$ if M admits an isometric immersion into $\bar{M}$ such that

$$J(T_m(M)) \subset T_m(M)^{\perp} ,$$

where $T_m(M)$ denotes the tangent space of M at m, and $T_m(M)^{\perp}$ the normal space at m. Denote by $\bar{M}^{n+p}(c)$ a $2(n+p)$-dimensional Kaehler manifold of constant holomorphic sectional curvature c. Let h be the second fundamental form of M in $\bar{M}$, and denote by S the square of the length of the second fundamental form h. When $p = 0$, Chen-Ogiue [2] proved

Theorem A. *Let M be an n-dimensional compact totally real minimal submanifold immersed in $\bar{M}^n(c)$. If*

$$S < \frac{n(n+1)}{4(2n-1)}c ,$$

then M is totally geodesic.

Theorem B. *Let M be an n-dimensional totally real minimal submanifold immersed in $\bar{M}^n(c)$. If the sectional curvature of M is constant, then M is either totally geodesic or has nonpositive sectional curvature. Moreover, if the second fundamental form of the immersiom is parallel, then M is totally geodesic or flat.*

Theorem B is a generalization of Houh's theorem [4]. Moreover, Ludden-Okumura-Yano [5] studied an n-dimensional totally real minimal submanifold M of CP^n satisfying

$$S = \frac{n(n+1)}{2n-1} , \tag{1.1}$$

where CP^n denotes an n-dimensional complex projective space of constant holomorphic sectional curvature 4, and gave an example of totally real

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minimal surface immersed in CP^2, which just satisfies the above condition (1.1). Let S^1 be a unit sphere of dimension 1. Then $S^1 \times S^1$ is a compact minimal totally real surface immersed in CP^2 with $S = 2$. Concerning this Ludden-Okumura-Yano [5] proved

Theorem C. *If M is a compact n-dimensional $(n > 1)$ minimal totally real submanifold of CP^n satisfying* (1.1), *then $n = 2$ and $M = S^1 \times S^1$.*

The purpose of this paper is to study a compact n-dimensional totally real submanifold M immersed in CP^n satisfying certain condition on the second fundamental form h of M, which reduces to condition (1.1) if M is minimally immersed in CP^n. Our method is based on that of Braidi-Hsiung [1].

2. Local formulas

Let $\bar{M}$ be a Kaehler manifold of dimension $2n$, and M an n-dimensional totally real submanifold immersed in $\bar{M}$. Choose a local field of orthonormal frames $e_1, \cdots, e_{2n}$ in $\bar{M}$ such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M (and hence the remaining vectors $e_{n+1}, \cdots, e_{2n}$ are normal to M). Unless stated otherwise, we shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C, \cdots \leq 2n\ , \quad 1 \leq i, j, k, \cdots \leq n\ , \quad n+1 \leq a, b, c, \cdots \leq 2n\ ,$$

and when a letter appears in any term as a subscript and a superscript, it is understood that this letter is summed over its range. Denote Je_i by e_{i^*} for $i = 1, \cdots, n$, and let $w^1, \cdots, w^{2n}$ be the field of dual frames with respect to the frame field of $\bar{M}$ chosen above. Then the structure equations of $\bar{M}$ are

$$dw^A = -w^A_B \wedge w^B\ , \tag{2.1}$$

$$w^A_B + w^B_A = 0\ , \quad w^i_j = w^{i^*}_{j^*}\ , \quad w^{i^*}_j = w^{j^*}_i\ , \tag{2.2}$$

$$\begin{gathered} dw^A_B = -w^A_C \wedge w^C_B + \Phi^A_B\ , \qquad \Phi^A_B = \tfrac{1}{2} K^A_{BCD} w^C \wedge w^D\ , \\ K^A_{BCD} + K^A_{BDC} = 0\ . \end{gathered} \tag{2.3}$$

Restriction of these frames to M gives

$$w^a = 0\ . \tag{2.4}$$

Since $0 = dw^a = -w^a_i \wedge w^i$, by Cartan's lemma we may write

$$w^a_i = h^a_{ij} w^j\ , \qquad h^a_{ij} = h^a_{ji}\ , \tag{2.5}$$

and from (2.2) it follows that

$$h^{i^*}_{jk} = h^{j^*}_{ik}\ . \tag{2.6}$$

Using these formulas we obtain

$$dw^i = -w^i_j \wedge w^j , \qquad w^i_j + w^j_i = 0 , \tag{2.7}$$

$$dw^i_j = -w^i_k \wedge w^k_j + \Omega^i_j , \qquad \Omega^i_j = \tfrac{1}{2}R^i_{jkl}w^k \wedge w^l , \tag{2.8}$$

$$R^i_{jkl} = K^i_{jkl} + \sum_a (h^a_{ik}h^a_{jl} - h^a_{il}h^a_{jk}) . \tag{2.9}$$

The forms (w^i_j) define the Riemannian connection of M. We call $h^a_{ij}w^iw^je_a$ the second fundamental form of the immersion. Sometimes the second fundamental form is denoted by its components h^a_{ij}. $(\sum_i h^a_{ii}e_a)/n$ is called the mean curvature normal, and an immersion is said to be *minimal* if its mean curvature normal vanishes identically, i.e., if $\sum_i h^a_{ii} = 0$ for all a. Define the covariant derivative h^a_{ijk} of h^a_{ij}, h^a_{ijkl} and the Laplacian Δh^a_{ij} of the second fundamental form h^a_{ij} respectively by

$$h^a_{ijk}w^k = dh^a_{ij} - h^a_{il}w^l_j - h^a_{lj}w^l_i + h^b_{ij}w^a_b , \tag{2.10}$$

$$h^a_{ijkl}w^l = dh^a_{ijk} - h^a_{ljk}w^l_i - h^a_{ilk}w^l_j - h^a_{ijl}w^l_k + h^b_{ijk}w^a_b , \tag{2.11}$$

$$\Delta h^a_{ij} = \sum_k h^a_{ijkk} . \tag{2.12}$$

If $\bar{M}$ is locally symmetric, then we have the following equation (Braidi-Hsiung [1, p. 238]):

$$\begin{aligned}\sum_{a,i,j} h^a_{ij}\Delta h^a_{ij} = \sum_{a,i,j,k} &(h^a_{ij}h^a_{kkij} - K^a_{ijb}h^a_{ij}h^b_{kk} + 4K^a_{bki}h^b_{jk}h^a_{ij} \\ &- K^a_{kbk}h^a_{ij}h^b_{ij} + 2K^m_{kik}h^a_{mj}h^a_{ij} + 2K^m_{ijk}h^a_{mk}h^a_{ij}) \\ - \sum_{a,b,i,j,k,l} &[(h^a_{ik}h^b_{jk} - h^a_{jk}h^b_{ik})(h^a_{il}h^b_{jl} - h^a_{jl}h^b_{il}) \\ &+ h^a_{ij}h^a_{kl}h^b_{ij}h^b_{kl} - h^a_{ij}h^a_{ki}h^b_{kj}h^b_{ll}] .\end{aligned} \tag{2.13}$$

3. Integral formulas

In this section we assume that $\bar{M}$ is a Kaehler manifold of dimension $2n$ and constant holomorphic sectional curvature c. Then the curvature tensor of $\bar{M}$ is given by

$$K^A_{BCD} = \tfrac{1}{4}c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}) , \tag{3.1}$$

where δ_{AC} denotes the Kronecker deltas. Let M be an n-dimensional totally real submanifold immersed in $\bar{M}^n(c)$. From the condition on the dimensions of M and $\bar{M}$ it follows that $e_{1*}, \cdots, e_{n*}$ is a frame for $T_m(M)^\perp$. Noticing this and using (2.6) and (3.1) we can reduce (2.13) to

$$\sum_{a,i,j} h^a_{ij}\Delta h^a_{ij} = \sum_{a,i,j,k} h^a_{ij}h^a_{kkij} + \tfrac{1}{4}(n+1)c \sum_{a,i,j} h^a_{ij}h^a_{ij} - \tfrac{1}{2}c \sum_a \Big(\sum_i h^a_{ii}\Big)^2$$

$$(3.2) \qquad + \sum_{a,b,i,j,k,l} (h_{ij}^a h_{jk}^b h_{ki}^a h_{il}^b - h_{ij}^a h_{ij}^b h_{kl}^a h_{kl}^b) - \sum_{a,b,i,j,k,l} (h_{ik}^a h_{kj}^b - h_{ik}^b h_{kj}^a)(h_{il}^a h_{lj}^b - h_{il}^b h_{lj}^a) .$$

For each a, let H_a denote the symmetric matrix (h_{ij}^a). Then (3.2) can be written as

$$\begin{aligned}\sum_{a,i,j} h_{ij}^a \Delta h_{ij}^a = &\sum_{a,i,j,k} h_{ij}^a h_{kkij}^a + \sum_a [\tfrac{1}{4}(n+1)c \operatorname{Tr} H_a^2 - \tfrac{1}{2}c(\operatorname{Tr} H_a)^2] \\ (3.3) \qquad &+ \sum_{a,b} \{\operatorname{Tr}(H_a H_b - H_b H_a)^2 - [\operatorname{Tr}(H_a H_b)]^2 \\ &+ \operatorname{Tr} H_b \operatorname{Tr}(H_a H_b H_a)\} ,\end{aligned}$$

where $\operatorname{Tr} H_a^2$ denotes the trace of the matrix H_a^2. (3.3) was obtained by Chen-Ogiue [2] for a totally real minimal submanifold M^n immersed in $\bar{M}^n(c)$. Now set

$$S_{ab} = \sum_{i,j} h_{ij}^a h_{ij}^b , \quad S_a = S_{aa} , \quad S = \sum_a S_a ,$$

so that S_{ab} is a symmetric $(n \times n)$-matrix and can be assumed to be diagonal for a suitable choice of $e_{n+1}, \cdots, e_{2n}$, and S is the square of the length of the second fundamental form h_{ij}^a of M. Since $\operatorname{Tr} A^2 = \sum_{i,j} (a_{ij})^2$ is independent of the choice of a frame, for any symmetric $A = (a_{ij})$ we can rewrite (3.3) as

$$\begin{aligned}\sum_{a,i,j} h_{ij}^a \Delta h_{ij}^a = &\sum_{a,i,j,k} h_{ij}^a h_{kkij}^a + \tfrac{1}{4}(n+1)cS - \sum_a S_a^2 \\ (3.4) \qquad &+ \sum_{a,b} \operatorname{Tr}(H_a H_b - H_b H_a)^2 - \tfrac{1}{2}c \sum_a (\operatorname{Tr} H_a)^2 \\ &+ \sum_{a,b} \operatorname{Tr} H_b \operatorname{Tr}(H_a H_b H_a) .\end{aligned}$$

For later development we need the following lemma (see [1] and [3]):

Lemma 1. *Let A and B be symmetric $(n \times n)$-matrices. Then*

$$-\operatorname{Tr}(AB - BA)^2 \le 2 \operatorname{Tr} A^2 \operatorname{Tr} B^2 ,$$

and the equality holds for nonzero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\bar{A}$ and $\bar{B}$ respectively, where

$$\bar{A} = \left[\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right] , \qquad \bar{B} = \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right] .$$

Moreover, if A_1, A_2, A_3 are symmetric $(n \times n)$-matrices such that

$$-\operatorname{Tr}(A_a A_b - A_b A_a)^2 = 2 \operatorname{Tr} A_a^2 \operatorname{Tr} A_b^2 , \quad 1 \le a, b \le 3 , \quad a \neq b ,$$

then at least one of the matrices A_a must be zero.

By applying Lemma 1 we obtain

$$(3.5)\qquad \begin{aligned} &-\sum_{a,b}\mathrm{Tr}(H_aH_b - H_bH_a)^2 + \sum_a S_a^2 - \frac{1}{4}(n+1)cS \\ &\le 2\sum_{a\neq b} S_aS_b + \sum_a S_a^2 - \frac{1}{4}(n+1)cS \\ &= \left[\left(2-\frac{1}{n}\right)S - \frac{1}{4}(n+1)c\right]S - \frac{1}{n}\sum_{a>b}(S_a - S_b)^2 , \end{aligned}$$

which, together with (3.4), implies

$$(3.6)\qquad -\sum_{a,i,j} h_{ij}^a \Delta h_{ij}^a \le W - \sum_{a,i,j,k} h_{ij}^a h_{kkij}^a ,$$

where we have put

$$(3.7)\qquad \begin{aligned} W = &\left[\left(2-\frac{1}{n}\right)S - \frac{1}{4}(n+1)c\right]S + \frac{1}{2}c\sum_a(\mathrm{Tr}\,H_a)^2 \\ &- \sum_{a,b}\mathrm{Tr}\,H_b\,\mathrm{Tr}(H_aH_bH_a) . \end{aligned}$$

Theorem 1. *Let M be an n-dimensional compact oriented totally real submanifold immersed in $\bar{M}^n(c)$. Then*

$$(3.8)\qquad \int_M \left[W - \sum_a(\mathrm{Tr}\,H_a)\Delta(\mathrm{Tr}\,H_a)\right] * 1 \ge 0 ,$$

where $*1$ denotes the volume element of M.

Proof. First we obtain

$$\int_M \sum_{a,i,j,k}(h_{ijk}^a)^2 * 1 = -\int_M \sum_{a,i,j} h_{ij}^a \Delta h_{ij}^a * 1 \ge 0 .$$

On the other hand, we have (Braidi-Hsiung [1, p. 241])

$$\int_M \sum_{a,i,j,k} h_{ij}^a h_{kkij}^a * 1 = \int_M \sum_a(\mathrm{Tr}\,H_a)\Delta(\mathrm{Tr}\,H_a) * 1 .$$

From these equations and (3.6) follows the inequality

$$(3.9)\qquad \int_M \left[W - \sum_a(\mathrm{Tr}\,H_a)\Delta(\mathrm{Tr}\,H_a)\right] * 1 \ge \int_M \sum_{a,i,j,k}(h_{ijk}^a)^2 * 1 \ge 0 ,$$

which is just (3.8).

As a special case of Theorem 1 we have the following theorem which was proved essentially by Chen-Ogiue [2].

Theorem 2. *Let M be an n-dimensional compact oriented totally real minimal submanifold immersed in $\bar{M}(c)$. Then*

$$\int_M \left[\left(2 - \frac{1}{n}\right)S - \frac{1}{4}(n+1)c\right]S * 1 \geq 0 . \tag{3.10}$$

4. Main theorems

In this section we assume that M is an n-dimensional compact oriented totally real submanifold immersed in $\overline{M}^n(c)$, $n > 1$, and that M is not totally geodesic in $\overline{M}$ but satisfies

$$\int_M \left[W - \sum_a (\operatorname{Tr} H_a)\Delta(\operatorname{Tr} H_a)\right] * 1 = 0 . \tag{4.1}$$

Then (3.9) implies that $h^a_{ijk} = 0$, i.e., the second fundamental form of M is covariant constant, so that $\Delta h^a_{ij} = 0$, and all terms on both sides of (3.6) vanish. It follows that inequalities (3.4) and (3.5) imply

$$\frac{1}{n} \sum_{a>b} (S_a - S_b)^2 = 0 , \tag{4.2}$$

$$-\operatorname{Tr}(H_a H_b - H_b H_a)^2 = 2 \operatorname{Tr} H_a^2 \operatorname{Tr} H_b^2 \tag{4.3}$$

for any $a \neq b$. Then by Lemma 1 we may assume that $H_a = 0$ for $a = n+3$, $\cdots, 2n$, which shows that $S_a = 0$ for $a = n+3, \cdots, 2n$. But by (4.2) we can see that $S_a = S_b$ for any a, b. Since M is not totally geodesic, $n = 2$ and therefore by using Lemma 1 we can assume that

$$H_{n+1} = \lambda\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad H_{n+2} = \mu\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4.4}$$

From this it follows that M is a minimal surface immersed in $\overline{M}^2(c)$. Since the second fundamental form h of M^2 is covariant constant, the sectional curvature of M^2 is constant and hence M^2 is flat by Theorem B. On the other hand, by using (2.10) we obtain

$$dh^a_{ij} = h^a_{il} w^l_j + h^a_{lj} w^l_i - h^b_{ij} w^a_b . \tag{4.5}$$

Setting $a = 3$, $i = 1$, $j = 2$, we see that $d\lambda = dh^3_{12} = 0$, which means that λ is constant. Similarly, setting $a = 4$ and $i = j = 1$, we see that μ is constant. By (4.2) we get $\lambda^2 = \mu^2$, and since $S = \frac{1}{2}c$ we have $\lambda^2 + \mu^2 = \frac{1}{4}c$ so that $\lambda^2 = \frac{1}{8}c$. Since M is not totally geodesic, we may assume that $c > 0$ and $-\lambda = \mu = \frac{1}{2}\sqrt{c/2}$. Then (2.5) and (4.4) imply

$$w^3_1 = \lambda w^2 , \quad w^3_2 = \lambda w^1 , \quad w^4_1 = \mu w^1 , \quad w^4_2 = -\mu w^2 .$$

On the other hand, setting $a = 3$, $i = j = 1$ in (4.5), we have $w^3_4 = (2\lambda/\mu)w^2_1 = 2w^1_2$. Hence we obtain the following

Theorem 3. *Let M be an n-dimensional compact oriented totally real submanifold immersed in $\bar{M}^n(c)$, $n > 1$, such that M is not totally geodesic but satisfies condition* (4.1). *Then M is a flat surface minimally immersed in $\bar{M}^2(c)$, and with respect to an adapted dual orthonormal frame field w^1, w^2, w^3, w^4, the connection form (w^A_B) of $\bar{M}^2(c)$, restricted to M, is given by*

$$\begin{bmatrix} 0 & w^1_2 & -\lambda w^2 & -\mu w^1 \\ -w^1_2 & 0 & -\lambda w^1 & \mu w^2 \\ \lambda w^2 & \lambda w^1 & 0 & 2w^1_2 \\ \mu w^1 & -\mu w^2 & -2w^1_2 & 0 \end{bmatrix}, \qquad -\lambda = \mu = \frac{1}{2}\sqrt{\frac{c}{2}}\,.$$

Now we take an n-dimensional complex projective space CP^n of constant holomorphic sectional curvature 4 as an ambient space. Then Theorem 3 implies

Theorem 4. *Let M be an n-dimensional compact oriented totally real submanifold immersed in CP^n, $n > 1$, such that M is not totally geodesic but satisfies condition* (4.1). *Then $n = 2$ and $M = S^1 \times S^1$.*

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